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Analytical modeling of dielectric polymer actuators

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Contract Number: W7707-1-0938

Contract Scientific Authority: J.P. Szabo, (902) 426-0550 ext 3427

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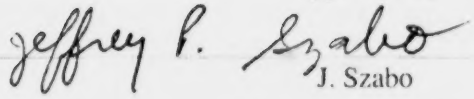

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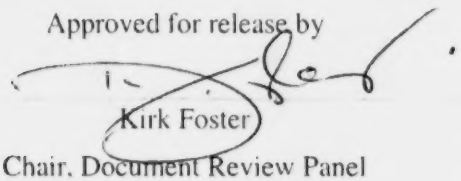

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Abstract

This report presents a literature review on the modeling of dielectric polymer actuators (DPA). The principle of actuation of these actuators and the underlying physics of dielectrics is exposed in order to introduce all the sources of difficulty responsible for the complex behaviour of DPAs. A variational formulation is proposed to model the behaviour of DPA, based on the variation of the dielectric enthalpy and on the virtual work of the non-conservative part of the electric displacement. A finite element method is developed, based on this formulation. A reverse mapping requirement is raised that limits the type of element that can be used. An example of a possible implementation is presented for the case of a linear tetrahedron. Finally, DPA modeling challenges are summarized and recommendations are made for future work.

Résumé

Ce rapport présente une revue de la littérature sur la modélisation des actionneurs à polymère diélectrique (APD). Le principe de fonctionnement de ces actionneurs, ainsi que les phénomènes physiques sous-jacents sont exposés afin de montrer toutes les sources de difficulté responsables du comportement complexe de ces APD. Une formulation variationnelle est proposée pour modéliser le comportement des APD, basée sur la variation de l'enthalpie diélectrique et sur le travail virtuel de la partie non conservative du déplacement électrique. Une méthode élément fini est développée, basée sur cette formulation. Un problème de réversibilité de la transformation est levé qui limite l'application de cette méthode à certains éléments. Un exemple possible d'implémentation est présenté pour le cas d'un tétraèdre linéaire. Enfin, les challenges de la modélisation d'un APD sont listés et des recommandations sont proposées pour de futurs travaux.

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Executive summary

Background

Dielectric Polymer Actuators (DPAs) that utilize the Maxwell stress effect consist of a thin membrane of non-conductive elastomer sandwiched between elastomeric electrodes. When an electrostatic charge is applied to the electrodes, the opposite charges attract, resulting in compression of the elastomer in one direction and extension in the other. Dielectric Polymers present a non-linear response to a time varying voltage due to: (i) the variable membrane thickness dependency of the electric field and (ii) the variable area of the electrodes. Maxwell stress acting on a compliant elastomer apparently generates the same effects as electrostriction (i. e. the dynamic response is roughly proportional to the square of the driving voltage). However it is clearly a different mechanism. It was demonstrated in the literature that DPAs have a great potential as "muscle-like actuators", and they present a good impedance match with water so that they could make good low frequency acoustic projectors. However, up to now, DPAs have been built by trial and error, using crude analytical models as design guidelines. Thus, there is a need for a more reliable design tool for developing optimized DPAs. Modeling and characterizing Dielectric Polymers (DP) present a lot of difficult challenges since these materials can present electric, dielectric and mechanical losses, as well non-linear material properties.

Principal Results

This work presents the state of the art in DPA modeling. It describes the underlying physics of DPAs to explain the origin of the major difficulties of DPA modeling and DP characterization. A finite element formulation was developed which could be the starting point for developing a finite element code for predicting the behaviour of DPAs. The background knowledge of this project is from the public domain. This knowledge has been enhanced by the finite element formulation proposed in this report.

Significance of Results

This work represents a significant milestone in the TIF project "Dielectric Actuators for Active/Passive Vibration Isolation". It lays the foundation for further modelling work on DPAs, which will be crucial for applications such as low frequency active sonar and active vibration isolation.

Future Work

Future modelling work will include (a) Exploring the effects of electrical and mechanical properties on actuation behaviour using simple models, (b) incorporating a DPA capability in the VVES suite of programs for vibration isolation, and (c) development of general finite element tools for DPAs.

Beslin, O. 2002. Analytical Modeling of Polymer Actuators. DRDC Atlantic CR 2002-160, MacDonald Dettwiler and Associates.

Sommaire

Contexte

Les actionneurs à polymère diélectrique (APD), dont le fonctionnement est basé sur l'effet de contrainte de Maxwell, sont constitués d'une mince membrane faite d'un élastomère non conducteur disposé entre des électrodes constituées, elles aussi, d'un élastomère. Lorsqu'une charge électrostatique est appliquée sur les électrodes, les charges opposées s'attirent et provoquent la compression de l'élastomère dans une direction et son allongement dans l'autre direction. Les polymères diélectriques ont une réponse non linéaire lorsqu'ils sont exposés à une tension qui varie avec le temps, et ce, pour les raisons suivantes : i) le champ électrique dépend de l'épaisseur de la membrane et ii) la surface des électrodes est variable. La contrainte de Maxwell qui s'exerce sur un élastomère souple produit apparemment des effets identiques à l'électrostriction (c.-à-d. que la réponse dynamique est approximativement proportionnelle au carré de la tension appliquée). Toutefois, il s'agit nettement d'un mécanisme différent. Selon la littérature, les APD présentent un potentiel élevé comme « actionneurs de type musculaire » et pourraient constituer de bons projecteurs acoustiques de faible fréquence en raison de leur impédance qui est semblable à celle de l'eau. Toutefois, jusqu'ici, on a élaboré des APD par approximation, en se basant, pour la conception, sur des modèles analytiques brutes. Il y a donc lieu de disposer d'un outil de conception plus fiable pour mettre au point des APD optimisés. La modélisation et la caractérisation des polymères diélectriques comportent de nombreux défis difficiles, car ces matériaux peuvent présenter des pertes mécaniques, diélectriques et électriques, ainsi que des propriétés de matériaux non linéaires.

Principaux résultats

On présente ici la fine pointe de la technologie en matière de modélisation des APD. On décrit les propriétés physiques sous-jacentes des APD pour expliquer l'origine des principales difficultés que comportent la modélisation des ADP et la caractérisation des polymères diélectriques. On a élaboré une formulation par éléments finis, qui pourrait constituer le point de départ de la mise au point d'un code d'éléments finis pour prévoir le comportement des APD. Les connaissances nécessaires à la mise en œuvre de ce projet sont du domaine public. Elles ont été approfondies grâce à la formulation par éléments finis proposée dans le présent rapport.

Importance des résultats

Ces travaux constituent une importante étape du projet financé dans le cadre du FIT intitulé « Actionneurs diélectriques d'isolation active et passive des vibrations ». Ils permettent d'établir une base à partir de laquelle seront effectués d'autres travaux de modélisation des APD, qui seront d'une importance cruciale pour des applications comme les sonars actifs à basse fréquence et l'isolation active des vibrations.

Beslin, O. 2002. Modélisation analytique d'actionneurs à polymère. RDDC Atlantique CR 2002-160, MacDonald Dettwiler and Associates.

Futurs travaux

Les travaux ultérieurs de modélisation comprendront : a) une étude des effets des propriétés électriques et mécaniques sur l'action des actionneurs, à l'aide de modèles simples; b) l'incorporation d'une capacité d'APD dans la suite de programmes VVES sur l'isolation des vibrations; et c) l'élaboration d'outils généraux d'éléments finis pour APD.

Beslin O. . 2002. Analytical Modeling of Polymer Actuators. DRDC Atlantic CR 2002-160, MacDonald Dettwiler and Associates.

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1. Introduction

The possible use of a dielectric polymer (DP) as a means of actuation was initially proposed by SRI International in 1998 [1]. The principle of actuation of a Dielectric Polymer Actuator (DPA) can be summarized as follows: A DPA is a condenser filled with a polymeric dielectric. The electrodes of the condenser are mechanically supported by the dielectric medium instead of being maintained by a rigid frame as classical condensers. When an electric voltage is applied to the condenser, the Maxwell electrostatic forces acting on the electrodes tend to squeeze the dielectric medium. The electrodes move toward one another until equilibrium is reached between the elastic stress of the dielectric medium and the Maxwell forces. When the dielectric is squeezed, depending on its Poisson ratio and depending on the compliance of the electrodes, it tends to expand along the directions of the plane of the condenser. This type of actuator benefits from two mechanisms of "amplification": The first mechanism is related to the compliance of the dielectric (*i*) When the electrodes tend to merge, the gap of the condenser decreases so that the electric field increases. (*ii*) If the bulk modulus of the dielectric is high, and if the electrodes are compliant, the dielectric and the electrodes expand, increasing the surface area of the condenser. Thus, both mechanisms tend to increase the capacity of the condenser when a voltage is applied to it. This coupling between the Maxwell stress and the elastic stress leads to a non linear mechanism of actuation, yielding a strain approximately proportional to the square of the driving voltage. In this description of the principle of actuation of a DPA, the permittivity of the dielectric was assumed to be a constant. However, in practice, other effects can interfere (in an additive or subtractive manner) with these two "amplification mechanisms":

- **Electrostriction: *i.e.* The permittivity of the dielectric medium is strain dependent.**
- **Quadratic permittivity *i.e.* The permittivity of the dielectric medium depends on the electric field.**

These two non-linear effects add some complexity to the development of a DPA prediction code, moreover, they present difficult challenges for characterizing the mechanical and dielectric properties of the dielectric polymer.

An additional source of difficulty when modeling and/or characterizing dielectric polymers is losses. Losses can be classified as electric, dielectric and mechanical.

- **Electric losses:**
- **Electric losses are due to mobile charges in the dielectric, providing a reminiscent conductivity, they can also come from the electrodes resistivity. Moreover, this resistivity can depend on the electrodes' expansion.**
- **Dielectric losses:**

- **Dielectric losses are due to the fact that the polarization vector is not instantaneously following the electric field in the dielectric medium. Complex relaxation mechanisms can occur, as well as phase transitions.**

- **Mechanical losses:**

- **Silicone elastomers can be considered as fully elastic, however, acrylate elastomers can present a marked viscoelastic behavior.**

All these losses contribute to a temperature increase, which is another source of variation of the permittivity. In particular, for polar rubbers, the polarization is highly temperature dependent, as rendered by the Debye model presented in section 2.4. Moreover, thermal runaway can occur in actuators made from elastomers with a reminiscent conductivity.

Several papers [1-4] from SRI International have experimentally demonstrated the efficiency and the potential of DPA using two families of dielectric polymers:

- **Silicone rubber films (polar rubber with a polydimethyl siloxane backbone)**

- **Polyacrylates (3M Company VHB acrylic series)**

Silicone rubbers are low-loss and soft polymers (0.2 to 0.4 MPa) with a fast response (dynamic response up to 1000Hz) and a good tear resistance. Polyacrylates present higher losses and a higher modulus (0.5-0.6MPa) with a slower response (up to 100 Hz). These papers have demonstrated the viability of the concept. However, further work is required to fully understand the underlying physics behind these results in order to predict the behavior of DPA.

A PhD study [5-7] was conducted at the Danish Polymer Center focused on the polyacrylate 3M VHB 4910. A simple analytical model (single rectangular layer) was developed to explain experimental measurements. It was found experimentally that the real part of the permittivity drops when the film is stretched. They tried to attribute this tendency to the electrostriction effect. However, the electrostrictive behavior of the VHB 4910 was revealed too weak to explain properly this experimental tendency. It was concluded that the discrepancies between the model and the experiment were probably due to the simplistic model they used. Particularly, a stretched rectangle is no longer a rectangle and some curving edge effects have to be accounted for. This study revealed that even for a simple experimental case, the associated prediction model must be sufficiently complex to take into account the geometrical effect of stretching.

The other big difficulty in DPA modeling is the availability of input parameters. As mentioned previously, the effects of intrinsic electrostriction and Maxwell stress merge to generate the same global behaviour. Thus, it is difficult to isolate each effect and to measure its associated coefficients, as the observed response is the result of a strong coupling between these different effects. Guillot et *al.*[8] have proposed a method to measure the electrostrictive coefficients of a polymer film without being influenced by Maxwell stress effects. They have used this method to characterize the electrostrictive coefficients of a polyurethane. This method should be tried to characterize the dielectric of a DPA. The weakness of the method,

however, is that the sample to be tested must be sandwiched between two silicone rubber layers. The stiffness of the rubber layers is assumed to be negligible compared to the stiffness of the sample. This assumption could probably not be made when characterizing silicone rubber films, however, it could be possible to characterize polyacrylates with this method.

In summary, modeling a DPA and characterizing a DP present a lot of challenges. The aim of this report was to give a synthesis on the state of the art in this domain and to provide the basis for the development of a finite element code for predicting the behaviour of DPAs.

Section 2 presents the underlying mechanism of dielectricity (polarization effect). The classical Debye model is presented. The Clausius Mossotti formula is established and used to explain the electrostriction effect and why uncompressible elastomers present a low electrostrictive effect.

Section 3 presents the Hamilton's functional of a DP (and a pre-stressed DP). The variation of the local dielectric enthalpy is calculated to show the origin of electrostriction and quadratic dielectricity. Dielectric loss is presented as the work of the non-conservative part of the electric displacement for a variation of the electric field. Finally, a finite element formulation for modeling Maxwell stress effect in DP is presented, calculating the variation of the dielectric enthalpy. A reverse mapping requirement issue is raised that limits the type of element that can be used. An example of possible implementation is presented for the case of a linear tetrahedron. Then, it is shown how the dielectric loss can be modeled by adding the virtual work of the non-conservative part of the electric displacement.

2. Summary of the theory of dielectrics

In this section, the theory of polarization is presented to explain the behaviour of a DP.

2.1 Vacuum condenser

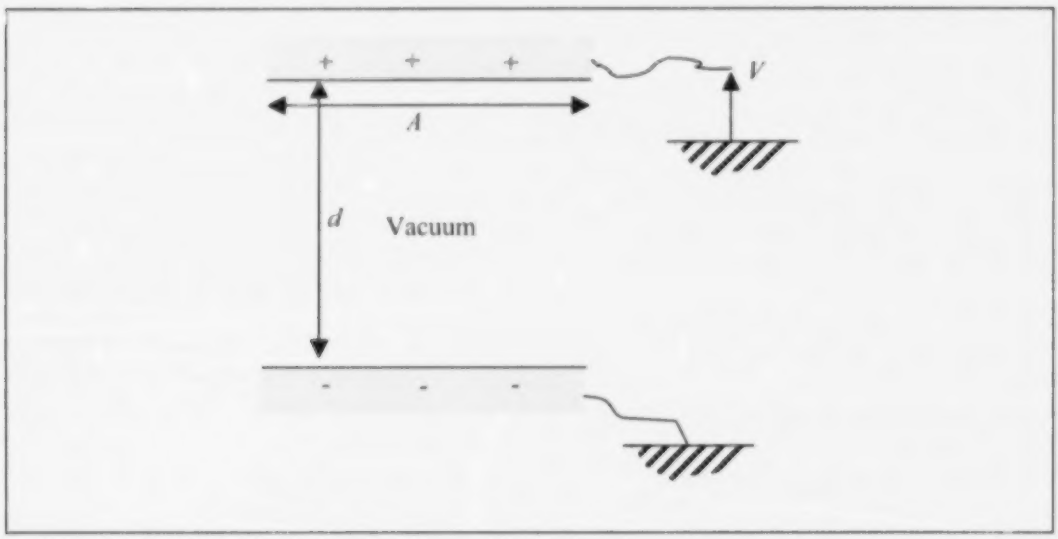

Figure 1: Vacuum condenser

Figure 1 represents a vacuum condenser consisting of two parallel plates (electrodes) of surface A. The distance between the two plates d is assumed to be small compared to the dimensions of the plates. If a potential V is imposed on the upper plate while the lower plate is left grounded, a surface density charge $+\sigma$ appears on the upper electrode while a surface density charge $-\sigma$ appears on the lower one. The charges, give rise to an electric field $\boldsymbol{E}$ which inside the condenser is practically homogeneous and directed perpendicular to the surface. Let us call z the axis perpendicular to the plates and oriented toward the up direction. The electric field is aligned along the z-axis and its z-axis component E_z is given by:

$$E_z = -\frac{V}{d}$$

The charge density on the electrodes σ, and the electric field between the electrodes $\boldsymbol{E}$, are classically related by the *permittivity of the vacuum* ε_0 as follows:

$$|\boldsymbol{E}| = \frac{\sigma}{\varepsilon_0}$$

The total electric charge maintained at potential V is $Q=\sigma A$.

The *capacity* of the condenser C is such that $Q=CV$, which can be expressed as:

$$C = \frac{\varepsilon_0 A}{d}$$

The condenser stores the total electrostatic energy U_e defined as:

$$U_e = \frac{1}{2}QV = \frac{1}{2}CV^2$$

Increasing the condenser capacity C allows one to increase the storable electrostatic energy. This can be achieved by changing the geometric parameters A and d and/or by introducing between the electrodes a medium having a permittivity higher than the permittivity of the vacuum. These kind of media are referred to as *dielectrics*.

2.2 Dielectric condenser

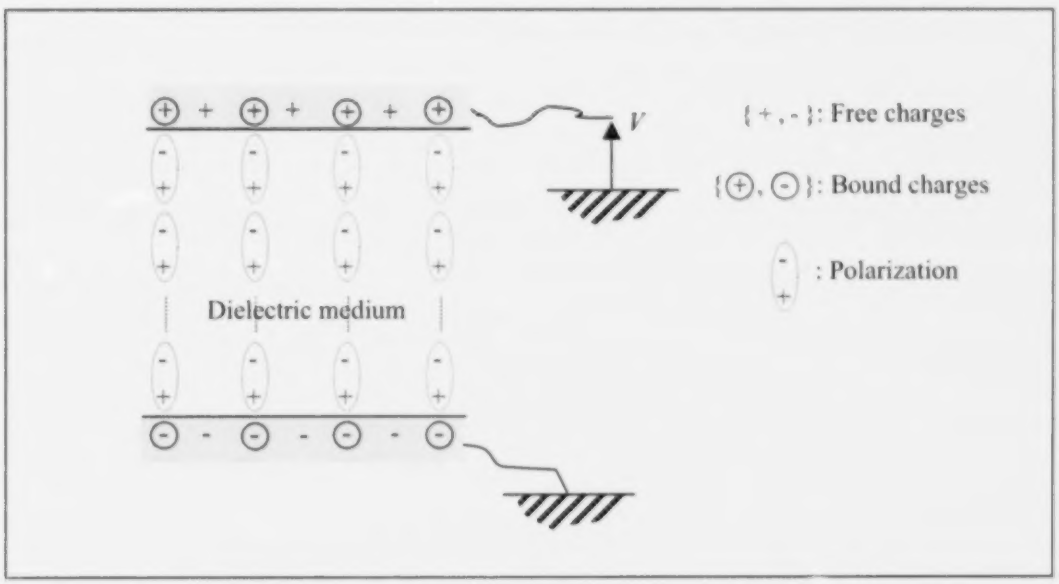

Figure 2: Dielectric condenser

Figure 2 represents the same condenser as the one introduced in the previous section but containing a dielectric medium between its electrodes instead of vacuum. If the same experiment is done on such a condenser, the electric potential and electric field (*i.e.* the

macroscopic electric field) will remain the same, however, the surface density charge σ will be greater than the one of the vacuum case. The increase in σ is due to the presence of *electric dipoles* $\{\boldsymbol{p}_k\}$ in the dielectric medium. Depending on the dielectric medium type, these dipoles can originate from different physical effects (electronic polarization, ionic polarization, polar molecules...). The average value of these electric dipoles over a small volume element is defined by the vector *polarization* $\boldsymbol{P}$. As shown in Figure 2, these electric dipoles attract on the electrodes, some additional charges, so called *bound charges*, , (*cf.* red circled + and – signs in Figure 2). The other charges, so called *free charges*, (*cf.* ordinary + and – signs in Figure 2) are the charges that would appear alone on the electrodes if the condenser was not filled with the dielectric medium. The sum of the free charges and bound charges is called the *true charges* and σ actually represents the surface density of these true charges.

Introducing the *permittivity of the dielectric medium* "ε", the charge density σ is related to the electric field as follows:

$$|\boldsymbol{E}| = \frac{\sigma}{\varepsilon}$$

As σ is higher for a dielectric medium than for vacuum, it can be seen that the dielectric permittivity is increased in the same proportion as for σ, since the electric field remains the same in both vacuum and dielectric-filled cases when the electric potential is imposed by an exterior generator.

2.3 Macroscopic fields in dielectrics

In a dielectric medium, it is convenient to introduce the *electric displacement field* $\boldsymbol{D}$ defined by:

Equation 1 $$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

Then, the basic equations that govern the behaviour of electric fields in dielectrics are:

Equation 2 $$\boldsymbol{E} = -\nabla\phi$$

Equation 3 $$\nabla \times \boldsymbol{E} = \boldsymbol{0}$$

Equation 4 $$\nabla \bullet \boldsymbol{D} = \rho$$

where ϕ is the electric potential and ρ is the electric charge density. $\boldsymbol{E}$ is referred to as the *macroscopic electric field.*

In the case of the dielectric condenser presented in Figure 2, the electric displacement can be considered as constant and collinear to the electric field (aligned along z-axis). Then, Equation

4 yields the simple relation between the surface density charge and the electric displacement: $D_z = \sigma$.

When the polarization is collinear to the electric field (most cases), Equation 1 can be turned into a simpler relation:

Equation 5 $$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

Sometimes, the *relative permittivity* ε_r is used, defined by: $\varepsilon = \varepsilon_0 \varepsilon_r$

Under the same assumption, the polarization can be related to the electric field by the relation:

Equation 6 $$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$$

where χ is the *electric susceptibility* which is related to the permittivity by: $\varepsilon_r = 1 + \chi$.

2.4 Theoretical models of the polarization vector

The polarization vector $\boldsymbol{P}$ was briefly introduced in section 2.2 as "the average value of electric dipoles over a small volume element". This section will present a summary of what are the origins of this polarization vector in dielectrics.
There is essentially six ways in which a material body can acquire a dielectric polarization. These ways are classified as:

- Pyroelectric effect (built-in permanent polarization).
- Ferroelectric effect (built-in reversible polarization).
- Piezoelectric effect (stress induced polarization).
- Electronic polarization (induced by a surrounding electric field acting on electrons/nucleus).
- Ionic polarization (induced by a surrounding electric field acting on ions).
- Dipolar polarization (induced by a surrounding electric field acting on polar molecules).

The material of interest for building dielectric polymer actuators must present a polarization induced by an exterior electric field, which corresponds to the last three effects listed above. *Electronic, ionic* and *dipolar polarizations* can be described (to the first order) using the *polarizability* parameter α. If n is the number of induced dipoles per unit volume, the polarization is expressed by:

$$\boldsymbol{P} = n\alpha \boldsymbol{E}_l$$

where $\boldsymbol{E}_l$ is the *local electric field*, which accounts for a correction due to the additional electric field induced by the polarization. A commonly accepted expression of the local field is:

Equation 7 $$\boldsymbol{E}_l = \boldsymbol{E} + \frac{\boldsymbol{P}}{3\varepsilon_0}$$

Using Equation 6 and Equation 7, the relative permittivity can be expressed by the following formula, classically referred to in literature as the *Clausius-Mossotti formula*:

Equation 8

$$\varepsilon_r = \frac{1 + 2(\frac{n\alpha}{3\varepsilon_0})}{1 - (\frac{n\alpha}{3\varepsilon_0})}$$

In Equation 8, α can stand for the *electronic polarization* ($n_e\alpha_e$), *ionic polarization* ($n_i\alpha_i$), *dipolar polarization* ($n_d\alpha_d$) or a mixture of them ($n\alpha = n_e\alpha_e + n_i\alpha_i + n_d\alpha_d$).

Polarizabilities are functions of frequency in the case of harmonic applied electric field, since the dipoles are not instantaneously following the driving electric field. Some relaxation and restoring force effects influence the magnitude and phase response of the polarization. Moreover, the dipolar polarization is greatly dependent on temperature. The Debye-Langevin theory gives (for low electric fields):

Equation 9

$$\alpha_d = \frac{p_0^2}{3kT}$$

where p_0 is the electric moment of the polar molecule, k is the *Boltzmann constant* and T is the temperature in degrees Kelvin. Equation 9 assumes that the dipoles are free to rotate and to follow instantaneously the driving electric field. In practice, additional restoring and friction forces must be considered depending of the type of molecules and geometry (polymer backbone orientation versus electric field direction).

Due to the various origins of polarization and due to relaxation processes, it is evident that Equation 6 (even assuming a frequency dependent susceptibility) is too simple to account for all the physical phenomena occurring with time varying electric field and dielectric thickness.

As a minimum, a time relaxation τ can be introduced in Equation 6, accounting for dielectric losses:

Equation 10

$$\boldsymbol{P} + \tau\dot{\boldsymbol{P}} = \varepsilon_0\chi\boldsymbol{E}$$

- **Digression on Electrostriction**

The Clausius-Mossotti formula given in Equation 8 shows that the permittivity of a dielectric must depend on the number of dipoles (and so molecules) per unit volume. This means that a non-constant volume deformation leads to changes in permittivity. This effect of strain dependent permittivity will be referred to as *electrostrictive effect* in the following sections. Moreover, if the main dielectric property is due to dipolar polarization, the permittivity depends on temperature through the polarizability α_d.

Remark: A formula describing the electrostrictive effect (to the first order) can be derived from Equation 8:

Equation 11

$$\varepsilon_r(\varepsilon_{ij}) \approx \frac{1+\frac{2}{3}\beta}{1-\beta} - \frac{\beta}{(1-\beta)^2}(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})$$

where $\beta = \frac{n\alpha}{\varepsilon_0}$

This last equation allows us to understand the origin of intrinsic electrostrictive effect: Any volume strain (through the "$\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$" term) will lead to a change in permittivity with a sensitivity that will depend on the parameter β.

3. Variational formulation

3.1 Hamilton's functional of a dielectric polymer

A dielectric polymer presents dielectric and mechanical losses associated with internal non-conservative forces (hysteresis). However, to clarify, the equations of motion of a dielectric polymer (DP) will be set up in this document using first the Hamilton principle of the associate conservative case and then, adding the virtual work of the non-conservative forces.

3.1.1 Description of the dielectric medium

Figure 3 presents the definition of the DP medium domain and variables.

Definitions of the variables are:

Ω and Γ : The volume and bounding surface, respectively, of the DP.

(x_1, x_2, x_3) : The co-ordinates of a DP material point.

u_i : The displacement in direction i of a DP material point.

φ: The electric potential in the DP.

E_i : The electric field in the DP.

D_i : The electric displacement field in the DP.

F_i : The external surface force distribution applied on the boundary of the DP.

Q : The external surface charge distribution applied on the boundary of the DP.

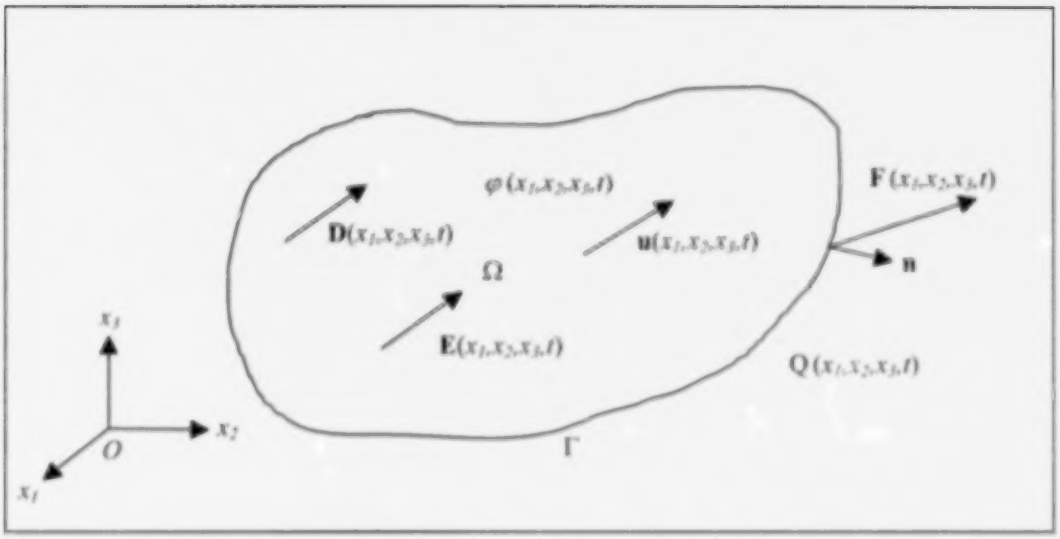

Figure 3. *Definition of the dielectric domain and fields*

3.1.2 Hamilton's functional

Considering $\{u_i,\varphi\}$ as the generalized displacement field, $\{\varepsilon_{ij}, E_i\}$ as the generalized strain field and $\{\sigma_{ij}, D_i\}$, as the generalized stress field, the DP Hamilton's functional can be written as:

$$H(t_1,t_2) = \int_{t_1}^{t_2}(T - V_{mech} - V_{elec} + W_{mech} + W_{elec})dt$$

Where the energy terms are:

$T = \frac{1}{2}\int_{\Omega} \rho \dot{u}_i \dot{u}_i \, d\Omega$:The Kinetic energy.

$V_{mech} = \frac{1}{2}\int_{\Omega} \sigma_{ij}\varepsilon_{ij} d\Omega$: The mechanical potential energy.

$V_{elec} = -\frac{1}{2}\int_{\Omega} D_i E_i d\Omega$: The dielectric enthalpy.

$W_{mech} = \int_{\Gamma} F_i u_i d\Gamma$: The work of the applied external surface mechanical loads.

$W_{elec} = \int_{\Gamma} Q\varphi \, d\Gamma$: The work of the applied external electric surface charges.

3.1.3 Variation of the dielectric enthalpy (conservative case)

The enthalpy per unit volume of a dielectric material is given by:

$$g = -\frac{1}{2} D_i E_i$$

The variation of the dielectric enthalpy associated with a virtual variation of the electric field is:

$$\delta g = -\frac{1}{2} D_i \delta E_i - \frac{1}{2} E_i \delta D_i$$

If hysteresis phenomena are neglected, the electric displacement is related to the electric field introducing the dielectric permittivity ξ. In the general case, this permittivity is a function of the mechanical strain and electric fields:

$$D_i = \xi(\varepsilon_{ij}, E_i) E_i$$

The variation of the electric displacement is then given by:

$$\delta D_i = \left(\xi + \left(\frac{\partial \xi}{\partial E_i} \right)_{\varepsilon_{ij}} E_i \right) \delta E_i + \left(\frac{\partial \xi}{\partial \varepsilon_{ij}} \right)_{E_i} E_i \delta \varepsilon_{ij}$$

Which leads to the following expression of the dielectric enthalpy variation:

Equation 12 $$\delta g = -\xi E_i \delta E_i - \frac{1}{2}\left(\frac{\partial \xi}{\partial E_i} \right)_{\varepsilon_{ij}} E_i^{\,2} \delta E_i - \frac{1}{2}\left(\frac{\partial \xi}{\partial \varepsilon_{ij}} \right)_{E_i} E_i^{\,2} \delta \varepsilon_{ij}$$

According to classical literature conventions [8], the three terms in the dielectric enthalpy variation of Equation 12 correspond to three physical effects, classified respectively by the following coefficient sets:

- ξ : The dielectric permittivity coefficient
- $\left(\frac{\partial \xi}{\partial E_i}\right)_{\varepsilon_{ij}}$: The quadratic dielectric coefficients
- $\frac{1}{2}\left(\frac{\partial \xi}{\partial \varepsilon_{ij}}\right)_{E_i}$: The electrostrictive stress coefficients

In the case of polyacrylate dielectric elastomers (3M VHB 4910) [6], it was shown that the electrostrictive effect was negligible. More generally, as shown previously in Equation 11, the change in permittivity is related to the volume strain in the polymer. Thus, the more the polymer is incompressible, the less it is electrostrictive. The variation of the permittivity with E (quadratic dielectric coefficient) is a second order dielectric effect.

3.1.4 Variation of the dielectric enthalpy (non-conservative case)

For a mono-dimensional problem, considering the principle of superposition, the general linear relation between the electric displacement field $D(t)$ and the time history of the electric field $E(t)$ can be expressed as[9]:

$$D(t) = \xi_\infty E(t) + \int_0^t E(u)\alpha(t-u)du$$

Where α(t) is a decay function, vanishing with t.

In this case, the generalized stress D(t) must be divided into its conservative and non-conservative parts:

$$D(t) = \xi_\infty E(t) + \tilde{D}(t)$$

With:

$$\tilde{D}(t) \equiv \int_0^t E(u)\alpha(t-u)du$$

The first term, proportional to E(t) corresponds to the conservative part of the electric displacement so that it can be expressed as the gradient of a potential (the dielectric enthalpy). This first part has been treated in the previous section.

The second term $\widetilde{D}(t)$ is related to the time history of the electric field. This hysteretic part is non-conservative (the work of this stress over a cycle is not zero). Rigorously, the use of Hamilton principle requires all the forces to be conservative. However, the equation of motion can be set up using first the Hamilton principle considering only the conservative case and then, the virtual work of the non-conservative forces can be added to the equations.

$$\delta W = -\delta g + \widetilde{D}\delta E$$

A numerical implementation of this non-conservative case is to consider a finite vector $\{\alpha_m\}$(m=0 to M) representing the sampling values of the function α(t) with a sampling period T.

Then the relation between D and E can be rewritten:

$$D_n = \alpha_0 E_n + \widetilde{D}_n$$

Where D_n and E_n are the sampled values: D_n=$D(nT)$ and E_n=$E(nT)$.

and where

$$\widetilde{D}_n \equiv \sum_{m=1}^{M} \alpha_m E_{n-m}$$

3.1.4.1 Measurement of $\alpha(t)$

The decay function α(t) is not easy to measure using time domain experiments. Dielectric analyzers provide the complex permittivity in frequency domain:

$$\xi^*(\omega) = \xi'(\omega) - i\xi''(\omega)$$

The relations between α(t) and $\xi^*(\omega)$ can be found classically in the literature [9] as:

$$\alpha(t) = \frac{2}{\pi}\int_0^{\infty}\left(\xi'(\omega) - \xi_{\infty}\right)\cos(\omega t)d\omega$$

or

$$\alpha(t) = \frac{2}{\pi}\int_0^{\infty}\xi''(\omega)\sin(\omega t)d\omega$$

Those two last expressions show that $\xi'(\omega)$ and $\xi''(\omega)$ are not independent. It can be shown[9] that the real and imaginary parts of the complex permittivity are interrelated by:

$$\xi'(\omega) = \xi_\infty + \frac{2}{\pi}\int_0^\infty \xi''(\mu)\frac{\mu}{\mu^2 - \omega^2}d\mu$$

or

$$\xi''(\omega) = \frac{2}{\pi}\int_0^\infty (\xi_1'(\mu) - \xi_\infty)\frac{\omega}{\omega^2 - \mu^2}d\mu$$

3.2 Hamilton's functional of a pre-stressed dielectric polymer.

3.2.1 Definitions of the pre-stressed state and associated static and dynamic fields

If the DP is pre-stressed by static electric and/or mechanical charge distributions $\{Q^0, \mathbf{F^0}\}$, the DP domain and its bounding surface change. The new domain and boundary surface are noted Ω^* and Γ^* respectively.

The following conventions are used:

The displacement field is expressed as:

$$u_i = u_i^0 + u_i^*$$

With:

u_i^0 : The constant displacement in direction i, induced by the static loads $\{Q^0, \mathbf{F^0}\}$.

u_i^* : The additional (time varying) displacement in direction i, induced by the dynamic loads $\{Q^*, \mathbf{F}^*\}$.

The strain field is expressed as:

$$\varepsilon_{ij} = \varepsilon_{ij}^0 + \varepsilon_{ij}^*$$

With:

ε_{ij}^0: The constant strain field, induced by the static loads $\{Q^0, \mathbf{F^0}\}$.

ε_{ij}^{*}: The additional (time varying) strain field, induced by the dynamic loads $\{Q^{*}, \mathbf{F}^{*}\}$.

The electric potential field is expressed as:

$$\varphi = \varphi^{0} + \varphi^{*}$$

with:

φ^{0} : The constant potential electric field, induced by the static loads $\{Q^{0}, \mathbf{F}^{\mathbf{0}}\}$.

φ^{*} : The additional (time varying) potential electric field, induced by the dynamic loads $\{Q^{*}, \mathbf{F}^{*}\}$.

The electric field is expressed as:

$$E_i = E_i^{0} + E_i^{*}$$

With:

E_i^{0}: The constant electric field, induced by the static loads $\{Q^{0}, \mathbf{F}^{\mathbf{0}}\}$.

E_i^{*}: The additional (time varying) electric field, induced by the dynamic loads $\{Q^{*}, \mathbf{F}^{*}\}$.

The electric displacement field is expressed as:

$$D_i = D_i^{0} + D_i^{*}$$

With:

D_i^{0}: The constant electric displacement field, induced by the static loads $\{Q^{0}, \mathbf{F}^{\mathbf{0}}\}$.

D_i^{*}: The additional (time varying) electric displacement field, induced by the dynamic loads $\{Q^{*}, \mathbf{F}^{*}\}$.

The mechanical stress field is expressed as:

$$\sigma_{ij} = \sigma_{ij}^{0} + \sigma_{ij}^{*}$$

With:

σ_{ij}^{0}: The constant mechanical stress field, induced by the static loads $\{Q^{0}, \mathbf{F}^{\mathbf{0}}\}$.

σ_{ij}^{*}: The additional (time varying) mechanical stress field, induced by the dynamic loads $\{Q^{*}, \mathbf{F}^{*}\}$.

3.2.2 Hamilton's functional

Using the notation presented in the previous section, the Hamilton's functional of a pre-stressed DP can be expressed as follow:

$$H^{*}(t_1, t_2) = \int_{t_1}^{t_2} \left(T - V_{mech} - V_{elec} + W_{mech}^{0} + W_{mech}^{*} + W_{elec}^{0} + W_{elec}^{*} \right) dt$$

With:

$T = \frac{1}{2} \int_{\Omega^{*}} \rho^{*} \dot{u}_i^{*} \dot{u}_i^{*} d\Omega$: The Kinetic energy.

$V_{mech} = \frac{1}{2} \int_{\Omega^{*}} (\sigma_{ij}^{0} + \sigma_{ij}^{*})(\varepsilon_{ij}^{0} + \varepsilon_{ij}^{*}) d\Omega$: The DP mechanical potential energy.

$V_{elec} = -\frac{1}{2} \int_{\Omega^{*}} (D_i^{0} + D_i^{*})(E_i^{0} + E_i^{*}) d\Omega$: The DP dielectric enthalpy.

$W_{mec}^{0} = \int_{\Gamma^{*}} F_i^{0} u_i^{*} d\Gamma$: The work of the applied static external surface mechanical loads.

$W_{mec}^{*} = \int_{\Gamma^{*}} F_i^{*} u_i^{*} d\Gamma$: The work of the applied dynamic external surface mechanical loads.

$W_{elec}^{0} = \int_{\Gamma^{*}} Q^{0} \varphi^{*} d\Gamma$: The work of the applied static external surface electric charges.

$W_{elec}^{*} = \int_{\Gamma^{*}} Q^{*} \varphi^{*} d\Gamma$: The work of the applied dynamic external surface electric charges.

3.3 Macroscopic electrostriction and finite element mapping

As mentioned previously, a dielectric polymer can present locally a negligible electrostrictive effect (particularly when its Poisson ratio is close to 0.5). However, when a finite element of DP is deformed (even if its volume is kept constant), the electric field inside the DP element can change due to its nodes' displacements (even if the electric potential values are kept constant at these nodes). Hence, the dielectric energy of a DP element can vary with strain, due to strain-induced variation of the electric field. If $g^{(e)}$ is the enthalpy of a DP element, and $\{u_i\}$ its nodal displacements, then an electrostatic stress tensor $\{X_i^{(e)}\}$ must exist so that:

$$X_i^{(e)} = -\frac{\partial g^{(e)}}{\partial u_i}$$

It will be shown in this section that the evaluation of this stress tensor using a finite element method requires an analytical expression of the transformation Jacobian matrix.

3.3.1 Mapping

The problem is discretized in space and time:

- The DP medium is subdivided into a finite number of elements "e".
- Time is discretized in samples n with a sampling period T.

Figure 4 presents a diagram of the mapping between the reference element *(ref)* and the real element *(e)* at two instants $t=(n-1)T$ and $t=nT$.

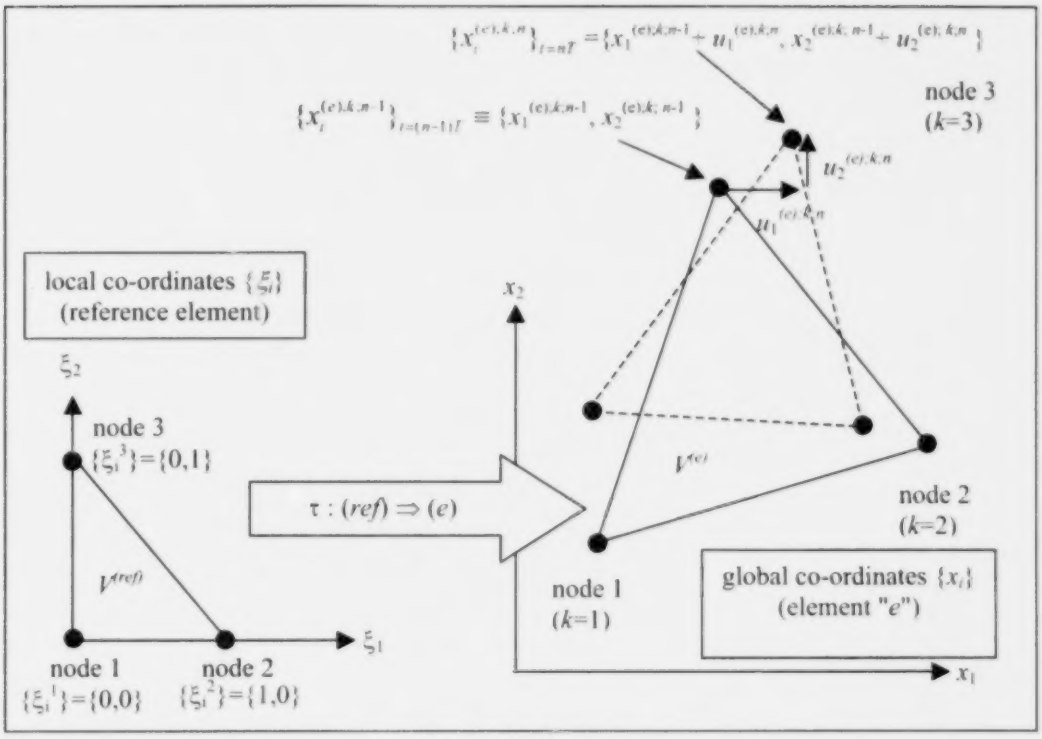

Figure 4. *Finite element mapping*

3.3.2 Generalized displacement field

The generalized displacement field at time $t=nT$ is expressed on one element e as:

$\varphi^{(e)}(x_i, nT) = \phi^{(e);k;n} N_k(x_i, \{u_i^{(e);k;n}\})$: The electric potential field

$u_i^{(e)}(x_i, nT) = u_i^{(e);k;n} N_k(x_i, \{u_i^{(e);k;n}\})$: The mechanical displacement field

Where:

- $\phi^{(e);k;n}$ is the value of the electric potential field on the node k of the element e at time nT
- $u_i^{(e);k;n}$ is the value of the mechanical displacement field on the node k of the element e at time nT
- $\{u_i^{(e);k;n}\}$ represents the set of the values of the mechanical displacement field at the nodes $\{k\}$ of the element e, at time nT.

- $N_k(x_i, \{u_i^{(e);k;n}\})$ is the interpolation function over the element (e) associated to the node k, and expressed as a function of the global co-ordinates $\{x_i\}$. It must be noted that those interpolation functions expressed in the global co-ordinates are functions of the nodal mechanical displacement set $\left\{u_i^{(e);k;n}\right\}$.

3.3.3 Dielectric enthalpy

Considering that the electric and electric displacement fields are co-linear and aligned along the axis 3 of the global coordinates, the dielectric enthalpy of the element e at time nT is given by:

$$g_n^{(e)} = \int_{V^{(e)}(\{u_i^{(e);k;n}\})} -\frac{1}{2} D^{(e)}(x_i, nT) E^{(e)}(x_i, nT) dV^{(e)}$$

Introducing the notation:

$$\widetilde{N}_k(x_i, \{u_i^{(e);k;n}\}) \equiv \frac{\partial N_k(x_i, \{u_i^{(e);k;n}\})}{\partial x_3}$$

The electric field can be expressed on the element (e) as:

$$E^{(e)}(x_i, nT) = -\phi^{(e);k;n} \widetilde{N}_k(x_i, \{u_i^{(e);k;n}\})$$

Considering first the conservative case (D proportional to E), the electric displacement is given by:

$$D^{(e)}(x_i, nT) = \alpha_0 E^{(e)}(x_i, nT)$$

The dielectric enthalpy can then be expressed as a function of the generalized displacements $\left\{u_i^{(e);k;n}, \phi^{(e);k;n}\right\}$:

Equation 13

$$g_n^{(e)} = -\frac{\alpha_0}{2} \phi^{(e);q;n} I^{kq}(\{u_i^{(e);k;n}\}) \phi^{(e);k;n}$$

Where:

$$I^{kq}(\{u_i^{(e);k;n}\}) = \int_{V^{(e)}(\{u_i^{(e);k;n}\})} \widetilde{N}_k(x_i, \{u_i^{(e);k;n}\}) \widetilde{N}_q(x_i, \{u_i^{(e);k;n}\}) dV^{(e)}$$

The $\{ I^{kq} \}$ integrals are symmetrical and can be calculated on the reference element as presented in Equation14:

Equation14

$$I^{kq}(\{u_i^{(e);k;n}\}) = \int_{V^{(ref)}} \left(\frac{\partial N_k}{\partial \xi_p} \frac{\partial \xi_p(\{x_i^{(e);k;n-1}\}, \{u_i^{(e);k;n}\})}{\partial x_3} \right) \left(\frac{\partial N_q}{\partial \xi_{p'}} \frac{\partial \xi_{p'}(\{x_i^{(e);k;n-1}\}, \{u_i^{(e);k;n}\})}{\partial x_3} \right) Det(J(\{x_i^{(e);k;n-1}\}, \{u_i^{(e);k;n}\})) dV^{(ref)}$$

Where

$Det(J(\{x_i^{(e);k;n-1}\}, \{u_i^{(e);k;n}\}))$ is the determinant of the Jacobian matrix of the transformation $\{\xi_i\} \Rightarrow \{x_i\}$ from the reference element to the element (e).

Equation 13 can be rewritten in term of a *capacity matrix* C^{qk}, as follows:

$$g_n^{(e)} = -\frac{1}{2} \phi^{(e);q;n} C^{qk}(\{x_i^{(e);k;n-1}\}, \{u_i^{(e);k;n}\}) \phi^{(e);k;n}$$

The capacity matrix depends on the position of the element nodes at time step *n-1* and on the displacement at time step *n*:

$$C^{qk}(\{x_i^{(e;k;n-1)}\}, \{u_i^{(e;k;n)}\}) = \alpha_0 \, I^{qk}(\{x_i^{(e;k;n-1)}\}, \{u_i^{(e;k;n)}\})$$

3.3.4 Variation of the dielectric enthalpy

The variation of the dielectric enthalpy is given by:

Equation 15

$$\delta g_n^{(e)} = -\delta \phi^{(e);q;n} C^{qk}(\{x_i^{(e);k;n-1}\}, \{u_i^{(e);k;n}\}) \phi^{(e);k;n} - \delta u_i^{(e);q';n} X_i^{(e);q';n}(\{x_i^{(e);k;n-1}\}, \{u_i^{(e);k;n}\})$$

where $\{ X_i^{(e);q';n} \}$ are the electrostatic Maxwell nodal forces given by.

$$X_i^{(e);q';n}(\{x_i^{(e);k;n-1}\}, \{u_i^{(e);k;n}\}) \equiv \frac{1}{2} \phi^{(e);q;n} \frac{\partial C^{qk}}{\partial u_i^{(e);q';n}} \phi^{(e);k;n}$$

Maxwell electrostatic forces are functions of the nodal electric potential as well as the derivatives of the matrix capacity versus the mechanical nodal displacements

3.3.5 Mapping requirements

To evaluate $\frac{\partial C^{qk}}{\partial u_i^{(e);q';n}}$, the analytical expressions of $\{\frac{\partial \xi_i}{\partial x_j}\}$ as functions of the global coordinates $\{x_i\}$ are required. Possible finite elements could be [10]:

- *Linear tetrahedron* (4 nodes, C^0) for 3D problems.
- *Linear Triangle* (3 nodes, C^0) for 2D problems)

3.3.5.1 Linear tetrahedron

Figure 5 presents a linear tetrahedron and Equation 16 presents its associated interpolation and mapping functions (isoparametric element):

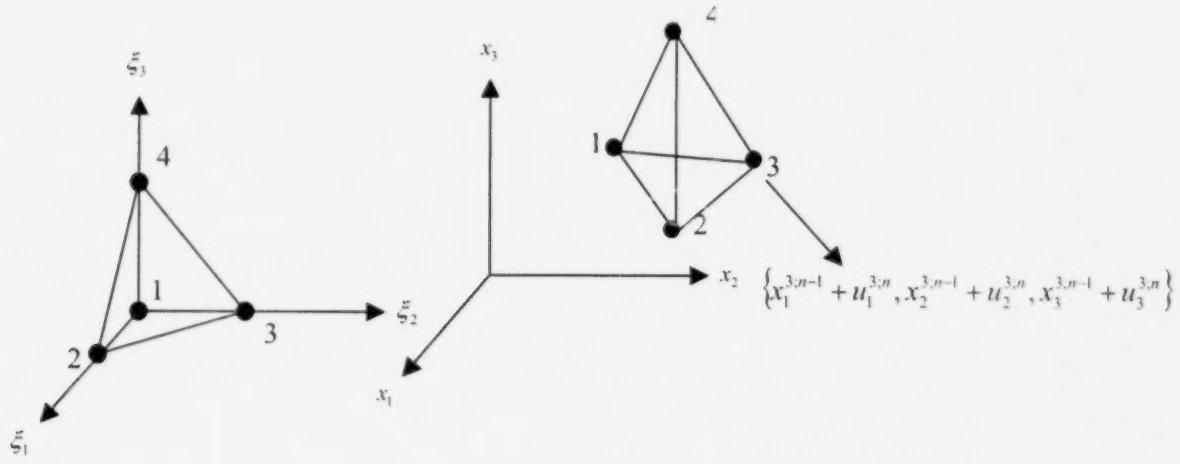

Figure 5: Linear tetrahedron

Equation 16: Linear tetrahedron interpolation and mapping functions:

$$N_1(\xi_i) \equiv 1 - \xi_1 - \xi_2 - \xi_3$$

$$N_2(\xi_i) \equiv \xi_1$$

$$N_3(\xi_i) \equiv \xi_2$$

$$N_4(\xi_i) \equiv \xi_3$$

The relation between the derivatives of the interpolation functions on the reference and real element is given by the Jacobian matrix J:

$$\{\partial_{\xi_i}\} = [J]\{\partial_{x_i}\}$$

In the case of the linear tetrahedron, J is given by:

$$[J] = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} = \begin{bmatrix} \tilde{x}_1^{2;n} - \tilde{x}_1^{1;n} & \tilde{x}_2^{2;n} - \tilde{x}_2^{1;n} & \tilde{x}_3^{2;n} - \tilde{x}_3^{1;n} \\ \tilde{x}_1^{2;n} - \tilde{x}_1^{1;n} & \tilde{x}_2^{3;n} - \tilde{x}_2^{1;n} & \tilde{x}_3^{3;n} - \tilde{x}_3^{1;n} \\ \tilde{x}_1^{2;n} - \tilde{x}_1^{1;n} & \tilde{x}_2^{4;n} - \tilde{x}_2^{1;n} & \tilde{x}_3^{4;n} - \tilde{x}_3^{1;n} \end{bmatrix}$$

where $\tilde{x}_i^{k;n}$ are the coordinates of the nodes of the real element at the time step n: $\tilde{x}_i^{k;n} = x_i^{k;n-1} + u_i^{k;n}$.

The I^{kq} integrals given in Equation14 can be evaluated since an analytical expression of $\frac{\partial \xi_i}{\partial x_j}$ can be derived by inverting the Jacobian matrix J.

$$[J]^{-1} = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_3}{\partial x_1} \\ \frac{\partial \xi_1}{\partial x_2} & \frac{\partial \xi_2}{\partial x_2} & \frac{\partial \xi_3}{\partial x_2} \\ \frac{\partial \xi_1}{\partial x_3} & \frac{\partial \xi_2}{\partial x_3} & \frac{\partial \xi_3}{\partial x_3} \end{bmatrix}$$

The inverse and the determinant of the Jacobian matrix are given by:

$$[J]^{-1} = \frac{1}{\det(J)} \begin{bmatrix} J_{22}J_{33} - J_{32}J_{23} & J_{13}J_{32} - J_{12}J_{33} & J_{12}J_{23} - J_{13}J_{22} \\ J_{31}J_{23} - J_{21}J_{33} & J_{11}J_{33} - J_{13}J_{31} & J_{21}J_{13} - J_{23}J_{11} \\ J_{21}J_{32} - J_{31}J_{22} & J_{12}J_{31} - J_{32}J_{11} & J_{11}J_{22} - J_{12}J_{21} \end{bmatrix}$$

$$\det(J) = J_{11}(J_{22}J_{33} - J_{32}J_{23}) + J_{12}(J_{31}J_{23} - J_{21}J_{33}) + J_{13}(J_{21}J_{32} - J_{31}J_{22})$$

Each J_{nm} terms are functions of the previous positions of the element nodes at time step n-1 : $x_i^{k;n-1}$ and of the displacement at current time step n: $u_i^{k;n}$.

Let us introduce the following notations:

$$F_{pp'}(\{x_i^{(e);k;n-1}\},\{u_i^{(e);k;n}\}) \equiv$$

$$\frac{\partial \xi_p(\{x_i^{(e);k;n-1}\},\{u_i^{(e);k;n}\})}{\partial x_3} \frac{\partial \xi_{p'}(\{x_i^{(e);k;n-1}\},\{u_i^{(e);k;n}\})}{\partial x_3} Det(J(\{x_i^{(e);k;n-1}\},\{u_i^{(e);k;n}\}))$$

and

$$\tilde{I}_{qkpp'} \equiv \int_{V^{(ref)}} \frac{\partial N_k}{\partial \xi_p} \frac{\partial N_q}{\partial \xi_{p'}} dV^{(ref)}$$

The terms $F_{pp'}$ are functions of the real element nodes positions at time step n-1 and of the nodal displacement at time step n.

Integrals $\tilde{I}_{qkpp'}$ are constant terms as they are defined on the reference element. For the linear tetrahedron, the terms $\frac{\partial N_i}{\partial \xi_j}$ are constants equal to 0,1 or –1, as:

$$\frac{\partial N_1}{\partial \xi_i} = -1 \text{, for } i\text{=1 to 3}$$

$$\frac{\partial N_{i+1}}{\partial \xi_i} = 1 \text{, for } i\text{=1 to,3}$$

The other terms are zeros.

Using these notations, the capacity matrix terms can be expressed as:

$$C^{qk}(\{x_i^{(e);k;n-1}\},\{u_i^{(e);k;n}\}) = \alpha_0 \widetilde{I}_{qkpp'} F_{pp'}(\{x_i^{(e);k;n-1}\},\{u_i^{(e);k;n}\})$$

So that the terms $\frac{\partial C^{qk}}{\partial u_i^{k'}}$ can be calculated as:

$$\frac{\partial C^{qk}}{\partial u_i^{k'}} = \alpha_0 \widetilde{I}_{qkpp'} \frac{\partial F_{pp'}}{\partial u_i^{k'}}(\{x_i^{(e);k;n-1}\},\{u_i^{(e);k;n}\})$$

3.3.6 Virtual work of the non conservative part of the electric displacement

The non-conservative part of the electric displacement can be expressed as a weighted summation of the time history of the electric field from time step n-*M* to time step *n*-1, following the element in its motion with time. Using a tetrahedral element, the electric field is constant in the whole element since it is calculated by differentiating linear interpolation functions. So the non-conservative part of the electric displacement of the element (e) can be expressed as a function of the element electric field value with time:

$$\widetilde{D}_n^{(e)} \equiv \sum_{m=1}^{M} \alpha_m E_{n-m}^{(e)}$$

The variation of the electric field related to the variations of the nodal electric potential and nodal displacement is given by:

$$\delta E^{(e)} = -\delta\phi^{(e);k;n} \widetilde{N}_k - \delta u_i^{(e);q;n} \frac{\partial \widetilde{N}_k}{\partial u_i^{(e);q;n}} \phi^{(e);k;n}$$

The virtual work of the non conservative part of the electric displacement on the element is given by:

$$\delta W = \int_{V^{(e)}} \widetilde{D}_n^{(e)} \delta E^{(e)} dV^{(e)}$$

This work can be divided into the following electric and mechanical works:

$$\delta W = \delta W_\phi + \delta W_u$$

The electric work can be expressed as:

Equation 17

$$\delta W_{\phi} = -\delta\phi^{(e);k;n}\widetilde{Q}_k^{(e);n}\widetilde{D}_n^{(e)}$$

where the integrals $\widetilde{Q}_k^{(e);n}$ are given by:

$$\widetilde{Q}_k^{(e);n} \equiv \int_{V^{(e)}} \widetilde{N}_k dV^{(e)}$$

The mechanical work can be expressed as:

Equation 18

$$\delta W_u = -\delta u_i^{(e);q;n}\widetilde{D}_n^{(e)}\widetilde{F}_i^{(e);qk;n}\phi^{(e);k;n}$$

where the integrals $\widetilde{F}_i^{(e);qk;n}$ are given by:

$$\widetilde{F}_i^{(e);qk;n} \equiv \int_{V^{(e)}} \frac{\partial \widetilde{N}_k}{\partial u_i^{(e);q;n}} dV^{(e)}$$

Integrals $\widetilde{Q}_k^{(e);n}$ and $\widetilde{F}_i^{(e);qk;n}$ can be calculated on the reference element using the same technique as the one presented in the previous section, using the analytical expression of the Jacobian matrix.

Equation 15, Equation 17 and Equation 18 yield the contribution of the electromechanical coupling between the electric potential field and the displacement field to the general problem. The contributions of the other terms can be found in the classical finite element literature.

Note: To simplify meshing, linear tetrahedrons can be assembled to build triangular prismatic elements, as shown in the annex A.

4. Conclusion and recommendations

This report has presented the multiple challenges of modeling a DPA and characterizing a DP. It was demonstrated in the literature that DPAs have great potential. However, up to now, they have been built by trial and error, using crude analytical models as design guidelines. Some discrepancies between experimental and theoretically expected results were left without any clear answer, due to the complex behaviour of a DP.

DPA modeling presents challenges at two different and complementary levels:

- **The experimental characterization of DP.**
- **The modeling itself.**

These challenges are due to the complex behaviour of DP which can be summarized as follows:

DPs are soft materials presenting large strain.

DPs present a strong coupling between the electric field and the mechanical deformation field.

DPs can present high mechanical losses (viscoelasticity), particularly the acrylic ones.

DPs present high dielectric losses (hysteresis cycle in D,E diagram)

DPs can present electric losses (reminiscent conductivity)

DPs can present a small electrostriction effect.

The principle of actuation of DPAs rely precisely on behaviours (1) and (2). That is, they are wanted behaviours. These behaviours are a first source of complexity for modeling DPAs since they require a non linear model that must be solved in the time domain.

Behaviours (3) to (5) are unwanted properties that will have to be minimized when developing new DP materials. These behaviours decrease the actuator efficiency. Moreover, they make difficult the experimental characterization of DPs. Particularly, due to the strong electromechanical coupling, it will be difficult to separate dielectric and mechanical losses, making difficult the measurement of the dielectric and mechanical relaxation functions.

Behaviour (6) can be a good property if its effect tends to add the DPA principle of actuation (*i. e.* Maxwell stress + compliance of the dielectric medium). However it will make difficult the experimental measurement of the permittivity of the DP since the electrostrictive effect and the DPA principle of actuation produce the same effect (strain proportional to the square of the driving voltage).

This study has presented a possible finite element method to model Maxwell stress in a DPA. This method requires the analytical expression of the Jacobian matrix of the transformation as a function of the nodes' co-ordinates on the real elements. It was shown that a linear tetrahedral element could be used. Even the dielectric losses could be calculated (provided that the relaxation function of the DP can be measured). It will be interesting to use this *linear 3D dielectric element* to simulate the experimental results presented by Kofod et *al.* [7]. In this case, the predicted strain should be closer to the experiment than the crude model they used.

To build a reliable finite element code for predicting DPAs behaviour, it will be wise to start with a low loss and low electrostrictive material. To validate the proposed element, a soft elastic and non-dielectric material could be used instead of a dielectric material. This will insure a constant permittivity, however the strain will be small so that high voltages will be required, hopefully a compromise could be found between excitation voltage and material elasticity so that a non-linear relation between excitation and strain could be measured.

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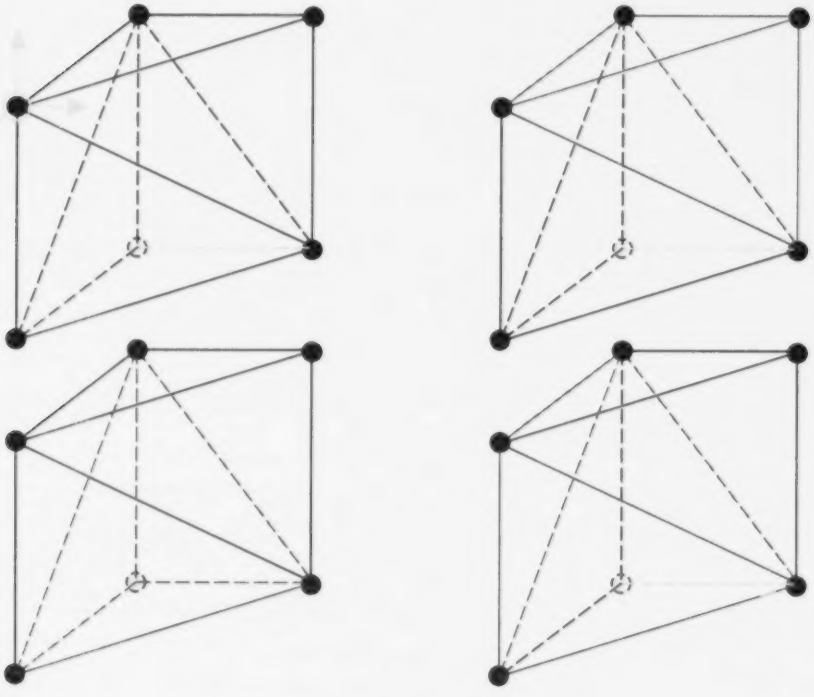
Annexe A: Example of tetrahedral elements assembly

Three tetrahedral elements can be assembled as follows, to make a one triangle prismatic element to simplify meshing (and post-processing data visualisation).



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DP	Dielectric Polymer
DPA	Dielectric Polymer Actuator

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This report presents a literature review on the modeling of dielectric polymer actuators (DPA). The principle of actuation of these actuators and the underlying physics of dielectrics is exposed in order to introduce all the sources of difficulty responsible for the complex behaviour of DPAs. A variational formulation is proposed to model the behaviour of DPA, based on the variation of the dielectric enthalpy and on the virtual work of the non conservative part of the electric displacement. A finite element method is developed, based on this formulation. A reverse mapping requirement issue is raised that limits the type of element that can be used. An example of possible implementation is presented for the case of a linear tetrahedron. Finally, DPA modeling challenges are summarized and recommendations are made for future work

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